Dynamical Systems (ME221)

Week 07: Inverse Laplace Transform

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Laplace Transform

- For a given function $f(t)$ with $f(t) = 0$ for $t < 0$, Laplace transform of this function is defined as follows:

$$\boxed{F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt}$$

s is a complex variable

$$s = a + jb$$

$$f(t) \xrightarrow{\quad\quad\mathcal{L}\quad\quad} F(s)$$

- From differential equations to algebraic equations

Laplace Transform Table

$\boldsymbol{x(t)}$	$\boldsymbol{X(s)}$
$\delta(t)$	1
$\varepsilon(t)$	$1/s$
$\varepsilon(t)t^n$	$n!/s^{n+1}$
$\varepsilon(t)e^{-\alpha t}$	$1/s+\alpha$
$\varepsilon(t)t^n e^{-\alpha t}$	$n!/(s+\alpha)^{n+1}$
$\varepsilon(t)\cos(\omega t)$	$s/s^2+\omega^2$
$\varepsilon(t)\sin(\omega t)$	$\omega/s^2+\omega^2$
$\varepsilon(t)e^{-\alpha t}\cos(\omega t)$	$(s+\alpha)/(s+\alpha)^2+\omega^2$
$\varepsilon(t)e^{-\alpha t}\sin(\omega t)$	$\omega/(s+\alpha)^2+\omega^2$

Laplace Transform Table

$x(t)$	$X(s)$	
$e^{-\alpha t}f(t)$	$F(s+\alpha)$	
$\varepsilon(t-\tau)f(t-\tau)$	$e^{-s\tau}F(s)$	
$f(t) * g(t)$	$F(s)G(s)$	
$\frac{d^n}{dt^n}f(t)$	$s^nF(s) - \sum_{k=1}^{n} s^{n-k}g_{k-1}$	$g_{k-1} = \left.\frac{d^{k-1}f}{dt^{k-1}}\right\|_{t=0}$
$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$	
$t^nf(t)$	$(-1)^n\frac{d^n}{ds^n}F(s)$	

Inverse Laplace Transform

$$f(t) = \mathcal{L}^{-1}[F(s)]$$

$$af_1(t) + bf_2(t) = \mathcal{L}^{-1}[aF_1(s) + bF_2(s)] = a\mathcal{L}^{-1}[F_1(s)] + b\mathcal{L}^{-1}[F_2(s)]$$

$$Y(s) = G(s)U(s)$$

- The objective is to find a projection from Laplace domain to the time domain
- There is no systematic method of determining inverse Laplace transform
- Derivative vs Integral analogy
- **Partial Fraction Expansion**

$$Y(s) \xrightarrow{\mathcal{L}^{-1}} y(t)$$

Partial Fraction Expansion

Re-write Y(s) as a linear combination of terms with known projections

$$Y(s) = \frac{d_q s^q + d_{q-1} s^{q-1} + \ldots + d_0}{s^p + c_{p-1} s^{p-1} + \ldots + c_0} = \frac{A_1}{s - s_1} + \frac{A_2}{s - s_2} + \ldots + \frac{A_p}{s - s_p}$$

$$(q \le p)$$

- The constants A_i (i = 1, 2, …, p) are called residues.
- How can we find the values of the residues?

Partial Fraction Expansion

- **CASE 1:** Distinct real roots

$$Y(s) = \frac{N(s)}{(s + r_1)(s + r_2) \cdots (s + r_n)}$$

$$Y(s) = \frac{A_1}{s + r_1} + \frac{A_2}{s + r_2} + \cdots + \frac{A_n}{s + r_n} \quad \text{where } A_i = \lim_{s \to -r_i} \{Y(s)(s + r_i)\}$$

Example

$$Y(s) = \frac{2(s+3)}{(s+1)(s+6)(s+2)}$$

Partial Fraction Expansion

$$Y(s) = \frac{A_1}{s+1} + \frac{A_2}{s+6} + \frac{A_3}{s+2}$$

$$A_1 = \lim_{s \to -1} \{Y(s)(s+1)\} = \lim_{s \to -1} \left\{ \frac{2(s+3)}{(s+2)(s+6)} \right\} = 0.8$$

$$A_2 = \lim_{s \to -6} \{Y(s)(s+6)\} = \lim_{s \to -6} \left\{ \frac{2(s+3)}{(s+1)(s+2)} \right\} = -0.3$$

$$A_3 = \lim_{s \to -2} \{Y(s)(s+2)\} = \lim_{s \to -2} \left\{ \frac{2(s+3)}{(s+1)(s+6)} \right\} = -0.5$$

Example

$$Y(s) = \frac{2(s+3)}{(s+1)(s+6)(s+2)}$$

Partial Fraction Expansion

$$Y(s) = \frac{0.8}{s+1} - \frac{0.3}{s+6} - \frac{0.5}{s+2}$$

$$\Downarrow$$

$$y(t) = 0.8e^{-t} - 0.3e^{-6t} - 0.5e^{-2t} \qquad t \geq 0$$

Example

- Direct calculation of residues

Partial Fraction Expansion

$$Y(s) = \frac{A_1}{s+1} + \frac{A_2}{s+6} + \frac{A_3}{s+2}$$

$$2(s+3) = A_1(s+6)(s+2) + A_2(\mathrm{s}+1)(\mathrm{s}+2) + A_3(s+1)(s+6)$$

$$A_1 + A_2 + A_3 = 0$$

$$8A_1 + 3A_2 + 7A_3 = 2$$

$$12A_1 + 2A_2 + 6A_3 = 6$$

$$\Rightarrow \quad A_1 = 0.8, A_2 = -0.3, A_3 = -0.5$$

Partial Fraction Expansion

- **CASE 2:** Distinct complex roots

$$Y(s) = \frac{s+1}{s(s^2+4s+5)}$$

- All coefficients are real numbers -> complex roots appear as conjugates

$$Y(s) = \frac{A_1}{s} + \frac{A_2 s + A_3}{s^2+4s+5}$$

$$(s+1) = A_1(s^2+4s+5) + (A_2 s + A_3)\text{s}$$

$$A_1 + A_2 = 0$$

$$4A_1 + A_3 = 1$$

$$5A_1 = 1$$

$$\Rightarrow \quad A_1 = 0.2, A_2 = -0.2, A_3 = 0.2$$

Partial Fraction Expansion

$$Y(s) = \frac{s+1}{s(s^2+4s+5)}$$

- All coefficients are real numbers → roots appear as complex conjugates

$$Y(s) = \frac{0.2}{s} + \frac{-0.2s+0.2}{s^2+4s+5} = \frac{0.2}{s} - \frac{0.2(s+2)}{(s+2)^2+1} + \frac{0.6}{(s+2)^2+1}$$

Note that $s^2 + 4s + 5 = (s+2)^2 + 1$

⇓

$$y(t) = 0.2\varepsilon(t) - 0.2e^{-2t}\cos t\,\varepsilon(t) + 0.6e^{-2t}\sin t\,\varepsilon(t)$$

$$y(t) = 0.2 - 0.2e^{-2t}\cos t + 0.6e^{-2t}\sin t \quad \text{for } t \geq 0$$

Partial Fraction Expansion

- **CASE 3:** Repeated real roots

$$Y(s) = \frac{N(s)}{(s+r_1)^p\left(s+r_{p+1}\right)\cdots(s+r_n)}$$

$$Y(s) = \frac{A_1}{(s+r_1)^p} + \frac{A_2}{(s+r_1)^{p-1}} + \cdots + \frac{A_p}{s+r_1} + \frac{A_{p+1}}{s+r_{p+1}} + \cdots + \frac{A_n}{s+r_n}$$

$$A_1 = \lim_{s\to -r_1}\{Y(s)(s+r_1)^p\}$$

$$A_2 = \lim_{s\to -r_1}\left\{\frac{d}{ds}[Y(s)(s+r_1)^p]\right\}$$

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$$A_i = \lim_{s\to -r_1}\left\{\frac{1}{(i-1)!}\frac{d^{i-1}}{ds^{i-1}}[Y(s)(s+r_1)^p]\right\} \qquad i = 1,2,\dots,p$$

$$A_i = \lim_{s\to -r_i}\{Y(s)(s+r_i)\}$$

$$i = p+1,\dots,n$$

Partial Fraction Expansion

- **CASE 3:** Repeated real roots

$$Y(s) = \frac{N(s)}{(s+r_1)^p\left(s+r_{p+1}\right)\cdots(s+r_n)}$$

$$Y(s) = \frac{A_1}{(s+r_1)^p} + \frac{A_2}{(s+r_1)^{p-1}} + \cdots + \frac{A_p}{s+r_1} + \frac{A_{p+1}}{s+r_{p+1}} + \cdots + \frac{A_n}{s+r_n}$$

$$y(t) = A_1 \frac{t^{p-1}}{(p-1)!} e^{-r_1 t} + A_2 \frac{t^{p-2}}{(p-2)!} e^{-r_1 t} + \cdots + A_p e^{-r_1 t} + A_{p+1} e^{-r_{p+1} t} + \cdots + A_n e^{-r_n t}$$

Example

$$Y(s) = \frac{s+1}{(s+2)^2(s+3)}$$

Partial Fraction Expansion

$$Y(s) = \frac{A_1}{(s+2)^2} + \frac{A_2}{s+2} + \frac{A_3}{s+3}$$

$$A_1 = \lim_{s \to -2} \{Y(s)(s+2)^2\} = \lim_{s \to -2} \left\{ \frac{(s+1)}{(s+3)} \right\} = -1$$

$$A_2 = \lim_{s \to -2} \left\{ \frac{d}{ds} [Y(s)(s+2)^2] \right\} = \lim_{s \to -2} \left\{ \frac{2}{(s+3)^2} \right\} = 2$$

$$A_3 = \lim_{s \to -3} \{Y(s)(s+3)\} = \lim_{s \to -3} \left\{ \frac{s+1}{(s+2)^2} \right\} = -2$$

Example

$$Y(s) = \frac{s+1}{(s+2)^2(s+3)}$$

Partial Fraction Expansion

$$Y(s) = -\frac{1}{(s+2)^2} + \frac{2}{s+2} - \frac{2}{s+3}$$

$$\Downarrow$$

$$y(t) = -te^{-2t} + 2e^{-2t} - 2e^{-3t} \text{ for } t \geq 0$$

Partial Fraction Expansion

What if q = p, that is to say numerator and denominator have the same order?

- Re-organize the expression as follows:

$$Y(s) = A + \frac{N(s)}{D(s)}$$

Example

$$Y(s) = \frac{s(s+3)}{(s+1)^2}$$

Partial Fraction Expansion

$$Y(s) = 1 + \frac{s-1}{(s+1)^2} = 1 + \frac{A_1}{(s+1)^2} + \frac{A_2}{s+1}$$

$$A_1 = \lim_{s\to-1}\{Y(s)(s+1)^2\} = \lim_{s\to-1}\{s-1\} = -2$$

$$A_2 = \lim_{s\to-1}\left\{\frac{d}{ds}[Y(s)(s+1)^2]\right\} = \lim_{s\to-1}\{1\} = 1$$

$$Y(s) = 1 - \frac{2}{(s+1)^2} + \frac{1}{s+1}$$

$$y(t) = \delta(t) - 2te^{-t} + e^{-t} \text{ for } t \geq 0$$

Example: Exponential terms

$$Y(s) = \frac{e^{-2s}}{s^2 + 4s + 5}$$

Inverse Laplace Transform

$$Y(s) = \frac{e^{-2s}}{s^2 + 4s + 5} = e^{-2s} Y_1(s)$$

$$Y_1(s) = \frac{1}{s^2 + 4s + 5} = \frac{1}{(s+2)^2 + 1} \xrightarrow{\mathcal{L}^{-1}} y_1(t) = e^{-2t} \sin t \quad t \geq 0$$

$$y(t) = y_1(t-2) = e^{-2(t-2)} \sin(t-2) \qquad t \geq 2$$

Example: LTI Systems

$$\ddot{y} + 2\dot{y} + 5y = 5u \qquad y(0) = \dot{y}(0) = 0$$

Calculate the unit step response

$$G(s) = \frac{5}{s^2 + 2s + 5} \qquad U(s) = \frac{1}{s}$$

$$Y(s) = \frac{5}{s(s^2 + 2s + 5)} = \frac{A}{s} + \frac{Bs + C}{(s+1)^2 + 2^2}$$

Example: LTI Systems

Partial fraction expansion

$$5 = A(s^2 + 2s + 5) + (Bs + C)s$$

$$
\begin{array}{llll}
s^2 : & 0 = A + B & & A = 1 \\
s^1 : & 0 = 2A + C & \rightarrow & B = -1 \\
s^0 : & 5 = 5A & & C = -2
\end{array}
$$

$$\frac{s+2}{(s+1)^2 + 2^2} = \frac{(s+1)}{(s+1)^2 + 2^2} + \frac{1}{2}\frac{2}{(s+1)^2 + 2^2}$$

Example: LTI Systems

Partial fraction expansion

$$Y(s) = \frac{1}{s} - \frac{s+1}{(s+1)^2 + 2^2} - \frac{1}{2}\frac{2}{(s+1)^2 + 2^2}$$

$$y(t) = 1 - e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right) \qquad t \geq 0$$

Linearization and Transfer Function

- What if the system is nonlinear?
- Transfer Function concept works for LTI Systems → Linearize around an equilibrium point

$$\ddot{x} + 2\dot{x} + 3x^2 = u \qquad x(0) = \dot{x}(0) = 0$$

$$\underline{\bar{u} = 0, \bar{x} = 0}$$

$$\ddot{x} + 2\dot{x} = u \qquad x(0) = \dot{x}(0) = 0$$

$$\frac{X(s)}{U(s)} = \frac{1}{s(s+2)}$$

Linearization and Transfer Function

- What if the system is nonlinear?
- Transfer Function concept works for LTI Systems → Linearize around an equilibrium point or **redefine variables**

$$\ddot{y} + 2\dot{y} + 3 = u \qquad y(0) = \dot{y}(0) = 0$$

$$\underline{\tilde{u}(t) = u(t) - 3}$$

$$\ddot{y} + 2\dot{y} = \tilde{u} \qquad y(0) = \dot{y}(0) = 0$$

$$\frac{Y(s)}{\tilde{U}(s)} = \frac{1}{s(s+2)}$$

Inverse Laplace Transform via Convolution

$$y(t) = u(t) * g(t) = g(t) * u(t) = \int_0^t u(\tau)g(t-\tau)d\tau$$

$$\mathcal{L}\{u(t) * g(t)\} = \mathcal{L}\{u(t)\}\mathcal{L}\{g(t)\}$$

$$\boxed{\mathcal{L}^{-1}\{U(s)G(s)\} = u(t) * g(t)}$$

Example

$$Y(s) = \frac{1}{(s^2+1)^2} = \frac{1}{s^2+1}\frac{1}{s^2+1}$$

Inverse Laplace Transform

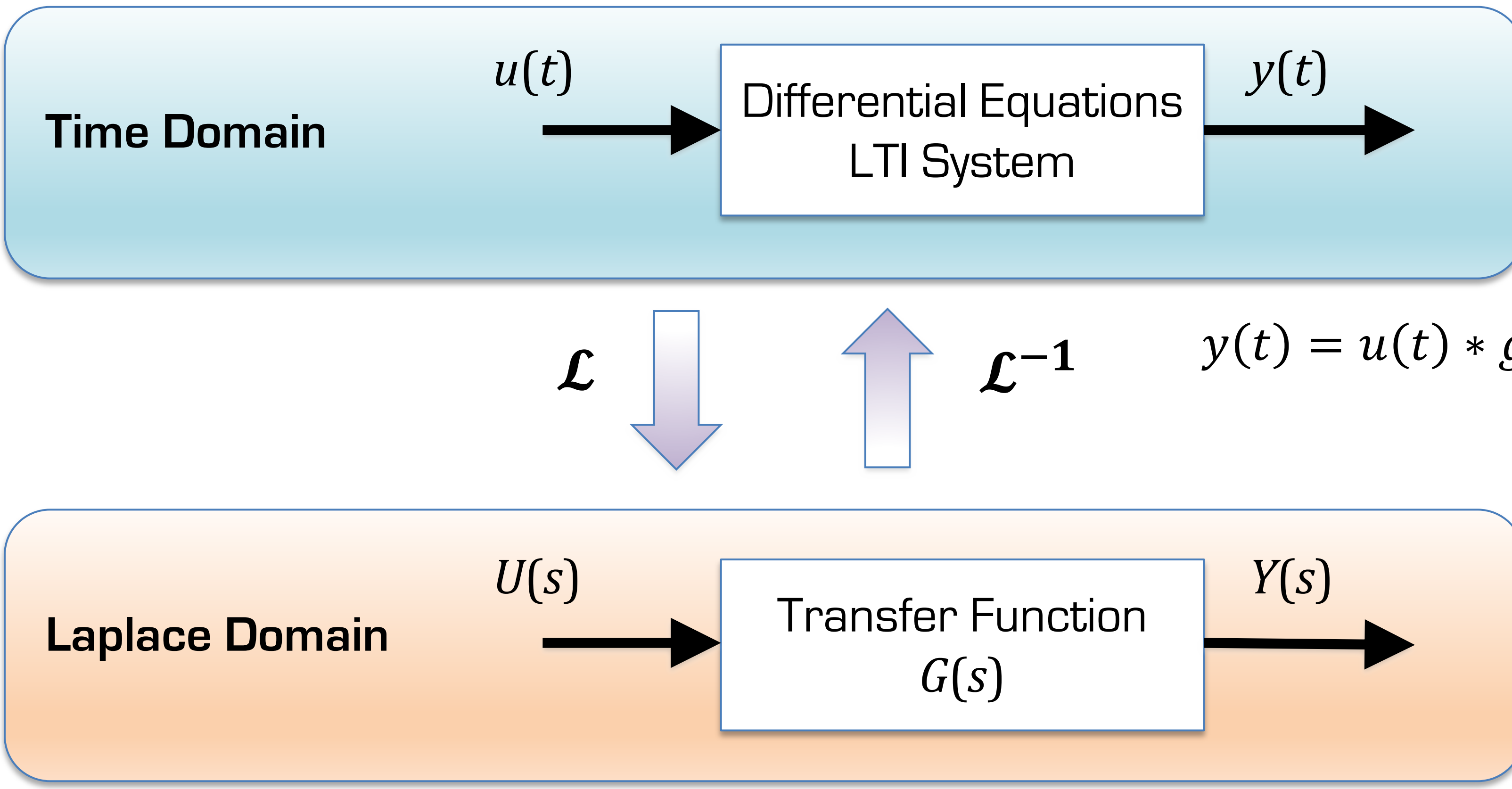
$$\mathcal{L}\{\sin(t)\varepsilon(t)\} = \frac{1}{s^2+1} \qquad \sin(a)\sin(b) = \frac{1}{2}[\cos(b-a) - \cos(b+a)]$$

$$y(t) = \sin(t)\varepsilon(t) * \sin(t)\varepsilon(t) = \int_0^t \sin(\tau)\sin(t-\tau)\,d\tau = \frac{1}{2}\int_0^t [\cos(2\tau - t) - \cos(t)]d\tau$$

$$= \frac{1}{2}\left[\frac{\sin(2\tau - t)}{2}\right]_0^t - \frac{1}{2}t\cos(t) \;= \frac{1}{2}\left[\frac{\sin(t)}{2} - \frac{\sin(-t)}{2}\right] - \frac{1}{2}t\cos(t)$$

$$= \frac{1}{2}\sin(t) - \frac{1}{2}t\cos(t)$$

Projection between Time and Laplace Domains



$$y(t) = u(t) * g(t)$$

$$Y(s) = U(s)G(s)$$